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The critical $A_{n-1}^{(1)}$ chain

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Abstract
We study the $A_{n-1}^{(1)}$ spin chain in the critical regime $|q| = 1$. We give free boson realizations of the vertex operators and their duals. Using these free boson realizations, we give integral representations for the correlation functions.

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1. Introduction

We shall consider a simple quantum mechanical model in one dimension, called the $A_{n-1}^{(1)}$ chain in the critical regime $|q| = 1$. In the earlier works [1, 2], the $A_{n-1}^{(1)}$ chain in the massive regime $-1 < q < 0$ was treated within the framework of the representation theory of the quantum affine algebra $U_q(\widehat{sl_n})$. In the context of representation theory, the correlation functions are realized as the trace of products of certain intertwiners (vertex operators) taken over the integrable highest-weight modules. As a result, correlation functions have been described by using the quantum Knizhnik–Zamolodchikov (qKZ) equations. For more general case of an *XYZ* chain, the free boson realization is not yet available (for recent developments in this direction see [3]). However, the correlation functions of the *XYZ* chain are related to the solutions of the qKZ equations, which are derived by the corner transfer matrix method (CTM) [4, 5]. The $A_1^{(1)}$ chain (*XXZ* chain) in the critical regime $|q| = 1$ was treated in [6, 7]. Though CTM fails to be well defined, the critical *XXZ* chain can be viewed as a limiting case of the *XYZ* chain, where the correlation function satisfies the qKZ equations. For the critical case $|q| = 1$, Jimbo and Miwa [7] presented integral representations of the correlation functions by solving the qKZ equations directly. In this connection we should mention the pioneering work by Smirnov [8]. Certain classes of solutions to the qKZ equations with $|q| = 1$ has been presented by Smirnov in connection with the form factors in the sine–Gordon theory. For the critical regime $|q| = 1$, the free boson realizations are available. Jimbo *et al* [6] gave the free boson realizations of the vertex operators of the critical *XXZ* chain, and presented certain integral representations of the correlation functions, which coincide with the formulae in [7]. In this connection we should mention Lukyanov's pioneering work [9]. Lukyanov constructed free boson realizations with an ultraviolet cut-off of the Zamolodchikov–Faddeev operators, and derived integral representations of the form factors for the sine–Gordon model and the $SU(2)$ invariant massive Thirring model. In this paper we shall consider a higher rank generalization of the critical *XXZ* chain. We present free boson realizations of vertex operators and derive integral representations for the correlation functions of the critical $A_{n-1}^{(1)}$ chain, which satisfies the qKZ equations, (1.5) and (1.6). For a higher-spin generalization of the critical *XXZ* chain, the free-field realizations of the type I vertex operators and integral representations of the correlation functions were presented in [10].

The critical $A_{n-1}^{(1)}$ chain is formulated as follows. Let $V = \oplus_{k=0}^{n-1}\mathbb{C}v_k$ be an n-dimensional vector space with the standard basis $v_0, v_1, \ldots, v_{n-1}$. Let V_k be a copy of V. The Hamiltonian $\mathcal{H}$ of the critical $A_{n-1}^{(1)}$ chain acting on $\otimes_{k\in\mathbb{Z}}V_k$ is given by

$$\mathcal{H} = \sum_{k=-\infty}^{\infty}\left\{q\sum_{\substack{a,b=0\\a>b}}^{n-1}E_{a,a}^{(k+1)}E_{b,b}^{(k)} + q^{-1}\sum_{\substack{a,b=0\\a<b}}^{n-1}E_{a,a}^{(k+1)}E_{b,b}^{(k)} - \sum_{\substack{a,b=0\\a\neq b}}^{n-1}E_{a,b}^{(k+1)}E_{b,a}^{(k)}\right\} \tag{1.1}$$

where the deformation parameter $|q| = 1$ and is generic. Here we have set the linear operator $E_{a,b} \in \mathrm{End}(V)$ by $E_{a,b}v_k = \sum_{j=0}^{n-1} v_j \left(E_{a,b}\right)_{j,k}$, where $\left(E_{a,b}\right)_{j,k} = \delta_{a,j}\delta_{b,k}$. We denote by $E_{a,b}^{(k)} \in \mathrm{End}(\otimes_{k\in\mathbb{Z}}V_k)$, the operator acting as $E_{a,b}$ on V_k and acting as identity elsewhere.

We study the correlation functions which describe the ground-state average $\langle\mathcal{O}\rangle$ of the local operator,

$$\mathcal{O} = E_{\epsilon_1,\epsilon_1'}^{(j_1)}E_{\epsilon_2,\epsilon_2'}^{(j_1+j_2)}\cdots E_{\epsilon_k,\epsilon_k'}^{(j_1+j_2+\cdots+j_k)}. \tag{1.2}$$

It is sufficient to consider the operator $\mathcal{O} = E_{\epsilon_1,\epsilon_1'}^{(1)}E_{\epsilon_2,\epsilon_2'}^{(2)}\cdots E_{\epsilon_N,\epsilon_N'}^{(N)}$, since the general case is reduced to this case by translation and taking linear combinations.

Let $V^* = \oplus_{k=0}^{n-1}\mathbb{C}v_k^*$ be the dual space of V, with the dual basis $v_0^*, v_1^*, \ldots, v_{n-1}^*$, i.e. $\langle v_j^*, v_k\rangle = \delta_{j,k}$. Let V_k^*, $(k \in \mathbb{Z})$ be a copy of V^*. The $2N$-point correlation function of the critical $A_{n-1}^{(1)}$ chain is a $(\otimes_{k=1}^N V_k^*) \otimes (\otimes_{k=1}^N V_{N+k})$-valued function depending on $2N$-spectral parameters $(\beta_1, \ldots, \beta_{2N})$,

$$\begin{aligned}G^{(N)}(\beta_1, \ldots, \beta_N|\beta_{N+1}, \ldots, \beta_{2N}) = &\sum_{\epsilon_1,\ldots,\epsilon_{2N}=0}^{n-1} v_{\epsilon_1}^* \otimes \cdots \otimes v_{\epsilon_N}^* \otimes v_{\epsilon_{N+1}} \otimes \cdots \otimes v_{\epsilon_{2N}}\\ &\times G^{(N)}(\beta_1, \ldots, \beta_N|\beta_{N+1}, \ldots, \beta_{2N})_{\epsilon_1,\ldots,\epsilon_N,\epsilon_{N+1},\ldots,\epsilon_{2N}}\end{aligned} \tag{1.3}$$

which are described by the following systems of difference equations (1.5)–(1.7). The ground state average of $\mathcal{O} = E_{\epsilon_1,\epsilon_1}^{(1)}\cdots E_{\epsilon_N,\epsilon_N}^{(N)}$ are obtained from the components $G^{(N)}$, by taking a shift parameter $\lambda = 2\pi$, and specializing the spectral parameters:

$$\langle E_{\epsilon_1,\epsilon_1}^{(1)}\cdots E_{\epsilon_N,\epsilon_N}^{(N)}\rangle = G^{(N)}(\beta+\pi\mathrm{i}, \ldots, \beta+\pi\mathrm{i}|\beta, \ldots, \beta)_{\epsilon_1\cdots\epsilon_N,\epsilon_N\cdots\epsilon_1}. \tag{1.4}$$

For the critical regime $|q| = 1$, CTM fails to be well defined. Nevertheless, this case can be viewed as a limiting case of Belavin's $\mathbb{Z}_n$ model [12], whose R-matrix satisfies the Yang–Baxter equation, the unitarity (the first inversion relation) and the crossing symmetry (the second inversion relation), so the correlation function $G^{(N)}(\beta_1, \ldots, \beta_N|\beta_{N+1}, \ldots, \beta_{2N})$ satisfies the quantum Knizhnik–Zamolodchikov equations (1.5), (1.6) and the normalization condition (1.7).

Quantum Knizhnik–Zamolodchikov equations

$$\begin{aligned}
&G^{(N)}(\beta_1,\ldots,\beta_j-\lambda\mathrm{i},\ldots,\beta_N|\beta_{N+1},\ldots,\beta_{2N})\\
&\qquad = R^{V^*V^*}_{j,j-1}(\beta_j-\beta_{j-1}-\lambda\mathrm{i})\cdots R^{V^*V^*}_{j,1}(\beta_j-\beta_1-\lambda\mathrm{i})\\
&\qquad\quad \times R^{V^*V}_{j,2N}(\beta_j-\beta_{2N})\cdots R^{V^*V}_{j,N+1}(\beta_j-\beta_{N+1})\\
&\qquad\quad \times R^{V^*V^*}_{j,N}(\beta_j-\beta_N)\cdots R^{V^*V^*}_{j,j+1}(\beta_j-\beta_{j+1})\\
&\qquad\quad \times G^{(N)}(\beta_1,\ldots,\beta_j,\ldots,\beta_N|\beta_{N+1},\ldots,\beta_{2N})
\end{aligned}\tag{1.5}$$

$$\begin{aligned}
&G^{(N)}(\beta_1,\ldots,\beta_N|\beta_{N+1},\ldots,\beta_j+\mathrm{i}\lambda,\ldots\beta_{2N})\\
&\qquad = R^{VV}_{j+1,j}(\beta_{j+1}-\beta_j+\lambda\mathrm{i})\cdots R^{VV}_{2N\cdot j}(\beta_{2N}-\beta_j+\lambda\mathrm{i})\\
&\qquad\quad \times R^{V^*V}_{1,j}(\beta_1-\beta_j)\cdots R^{V^*V}_{N,j}(\beta_N-\beta_j)\\
&\qquad\quad \times R^{VV}_{N+1,j}(\beta_{N+1}-\beta_j)\cdots R^{VV}_{j-1,j}(\beta_{j-1}-\beta_j)\\
&\qquad\quad \times G^{(N)}(\beta_1,\ldots,\beta_N|\beta_{N+1},\ldots,\beta_j,\ldots\beta_{2N}).
\end{aligned}\tag{1.6}$$

Normalization

$$\begin{aligned}
&G^{(N)}(\beta+\mathrm{i}(\pi-\lambda),\beta_2,\ldots,\beta_N|\beta_{N+1},\ldots,\beta_{2N-1},\beta)_{\epsilon_1\cdots\epsilon_N,\epsilon'_N\cdots\epsilon'_1}\\
&\qquad = \delta_{\epsilon_1,\epsilon'_1}G^{(N-1)}(\beta_2,\ldots,\beta_N|\beta_{N+1},\ldots,\beta_{2N-1})_{\epsilon_2\cdots\epsilon_N,\epsilon'_N\cdots\epsilon'_2}.
\end{aligned}\tag{1.7}$$

Here $R^{VV}_{ij}(\beta)\in\mathrm{End}\left((\otimes_{k=1}^N V_k^*)\otimes(\otimes_{k=1}^N V_{N+k})\right)$ signifies the matrix acting as $R^{VV}(\beta)\in\mathrm{End}(V\otimes V)$ on the (i,j)th tensor components and as identity elsewhere. Here $R^{VV}(\beta)\in\mathrm{End}(V\otimes V)$, $R^{V^*V^*}(\beta)\in\mathrm{End}(V^*\otimes V^*)$ and $R^{V^*V}(\beta)\in\mathrm{End}(V^*\otimes V)$ are given by (2.1), (3.4) and (3.6).

When we solve the qKZ equations (1.5) and (1.6), directly [7], we have a difficulty in that the solutions are determined only up to arbitrary periodic functions, so one has to single out in some way the correct solutions which correspond to the correlation functions. When we construct the solutions by the trace of the vertex operators, the ambiguity of the solutions is resolved. In this paper we give the free boson realizations of the type I vertex operators and their duals, and give the trace construction for the correlation functions of the critical $A^{(1)}_{n-1}$ spin. For the $n=2$ case, the vertex operators are self-dual, therefore the authors [6] did not have to consider the pair of vertex operators and their dual. For our higher-rank $n>2$ case under consideration, we need free boson realizations of the vertex operators and their 'dual'. This 'duality' problem is absent from the *XXZ* chain. In this connection, we should mention the work [11], in which the authors give the free boson realizations of the dual type II vertex operators of the $A^{(1)}_{n-1}$ affine Toda field theory with imaginary coupling.

Let us state the main result of this paper. We present integral representations of the correlation functions, $G^{(N)}(\beta_1,\ldots,\beta_N|\beta'_N,\ldots,\beta'_1)_{\epsilon_1\cdots\epsilon_N,\epsilon_N,\ldots,\epsilon_1}$, which obey the above equations (1.5)–(1.7),

$$\begin{aligned}
&G^{(N)}(\beta_1\cdots\beta_N|\beta'_N\cdots\beta'_1)_{\epsilon_1\cdots\epsilon_N,\epsilon_N,\ldots,\epsilon_1}\\
&\qquad = E_\lambda(\beta_1\cdots\beta_N|\beta'_N\cdots\beta'_1)\prod_{j,r}\int_{-\infty}^{\infty}\mathrm{d}\alpha_{j,r}\,K_\lambda(\{\alpha_{j,r}\})I_{\lambda;\epsilon_1\cdots\epsilon_N}(\{\alpha_{j,r}\}).
\end{aligned}\tag{1.8}$$

Here the factor $E_\lambda(\beta_1\cdots\beta_N|\beta'_N\cdots\beta'_1)$, the integral kernel $K_\lambda(\{\alpha_{j,r}\})$ and the integrand $I_{\lambda;\epsilon_1\cdots\epsilon_N}(\{\alpha_{j,r}\})$ are given in (6.29)–(6.31) respectively. The number of integrals is $(n-1)N$.

More detailed explanations of the notation are given in section 6. Integral representations (1.8) are available for arbitrary $\lambda > 0$. This formula has the ambiguity of a multiplicity constant.

In this paper we set the deformation parameter as $q = -\exp\left(\frac{\pi \mathrm{i}}{\xi}\right)$, where $\xi > 1$ and generic. Throughout this paper we restrict ourselves to the 'spin-0' case, i.e. we assume

$$G^{(N)}(\beta_1 \cdots \beta_N | \beta'_1 \cdots \beta'_N)_{\epsilon_1 \cdots \epsilon_N, \epsilon'_1 \cdots \epsilon'_N} = 0 \qquad \text{unless} \quad \{\epsilon_1, \ldots, \epsilon_N\} = \{\epsilon'_1, \ldots, \epsilon'_N\}. \tag{1.9}$$

Now we give a few words about the organization of the paper. In section 2 we introduce the R-matrix. In section 3 we give the characterizing relations of the vertex operators and their dual. In section 4 we give the free boson realizations of the vertex operators and their dual. In section 5 we give the proofs of the properties of the vertex operators. In section 6 we give the integral representations of the correlation functions. In appendix A we summarize the multi-gamma functions. In appendix B we summarize the normal ordering of the basic operators.

2. The critical $A_{n-1}^{(1)}$ chain

The purpose of this section is to introduce the R-matrix and to relate it to the Hamiltonian (1.1) of the present model.

2.1. R-matrix

Let $V = \oplus_{k=0}^{n-1} \mathbb{C} v_k$ be an n-dimensional vector space with the standard basis $v_0, v_1, \ldots, v_{n-1}$ Let us set the R-matrix $R^{VV}(\beta) \in \mathrm{End}(V \otimes V)$ by

$$R^{VV}(\beta) = r(\beta) \bar{R}(\beta) \qquad r(\beta) = -\frac{S_2(\mathrm{i}\beta | \frac{2\pi}{n}\xi, 2\pi) S_2(-\mathrm{i}\beta + \frac{2\pi}{n} | \frac{2\pi}{n}\xi, 2\pi)}{S_2(-\mathrm{i}\beta | \frac{2\pi}{n}\xi, 2\pi) S_2(\mathrm{i}\beta + \frac{2\pi}{n} | \frac{2\pi}{n}\xi, 2\pi)} \tag{2.1}$$

where $S_2(\beta|\omega_1, \omega_2)$ is the multi-sine function given in appendix A.

The auxiliary operator $\bar{R}(\beta) \in \mathrm{End}(V \otimes V)$ is given as follows:

$$\bar{R}(\beta) v_{k_1} \otimes v_{k_2} = \sum_{j_1, j_2 = 0}^{n-1} v_{j_1} \otimes v_{j_2} \bar{R}(\beta)_{j_1 j_2}^{k_1 k_2}. \tag{2.2}$$

Here the non-zero entries of the matrix elements $\bar{R}(\beta)_{j_1 j_2}^{k_1 k_2}$ are

$$\bar{R}(\beta)_{jj}^{jj} = 1 \tag{2.3}$$

$$\bar{R}(\beta)_{jk}^{jk} = -\frac{\mathrm{sh}\left(\frac{n}{2\xi}\beta\right)}{\mathrm{sh}\left(\frac{n}{2\xi}(\beta + \frac{2\pi \mathrm{i}}{n})\right)} \qquad (j \neq k) \tag{2.4}$$

$$\bar{R}(\beta)_{jk}^{kj} = \begin{cases} \dfrac{\mathrm{e}^{-\frac{n}{2\xi}\beta} \, \mathrm{sh}\left(\frac{\pi \mathrm{i}}{\xi}\right)}{\mathrm{sh}\left(\frac{n}{2\xi}(\beta + \frac{2\pi \mathrm{i}}{n})\right)} & (j < k) \\[2ex] \dfrac{\mathrm{e}^{\frac{n}{2\xi}\beta} \, \mathrm{sh}\left(\frac{\pi \mathrm{i}}{\xi}\right)}{\mathrm{sh}\left(\frac{n}{2\xi}(\beta + \frac{2\pi \mathrm{i}}{n})\right)} & (j > k). \end{cases} \tag{2.5}$$

Let V_k, $(k \in \mathbb{Z})$ be a copy of $V = \oplus_{k=0}^{n-1} \mathbb{C} v_k$.

The R-matrix $R^{VV}(\beta)$ satisfies the Yang–Baxter equations,

$$R_{12}^{VV}(\beta_1 - \beta_2) R_{13}^{VV}(\beta_1 - \beta_3) R_{23}^{VV}(\beta_2 - \beta_3) = R_{23}^{VV}(\beta_2 - \beta_3) R_{13}^{VV}(\beta_1 - \beta_3) R_{12}^{VV}(\beta_1 - \beta_2). \tag{2.6}$$

Here we denote by $R^{VV}_{jk}(\beta) \in \mathrm{End}(V_1 \otimes V_2 \otimes V_3)$, the operator acting as $R^{VV}(\beta)$ on the (j,k)th tensor components and as identity elsewhere.

The R-matrix $R^{VV}(\beta)$ satisfies the unitarity condition,

$$R^{VV}_{12}(\beta_1-\beta_2)R^{VV}_{21}(\beta_2-\beta_1)=\mathrm{id}. \tag{2.7}$$

2.2. *Transfer matrix*

In this subsection we introduce the transfer matrix $T(\beta) \in \mathrm{End}(\otimes_{k=1}^{N} V_k)$ and relate it to the Hamiltonian $\mathcal{H}$ (1.1) of the present model. Let us introduce the monodromy matrix $\mathcal{T}(\beta)$ acting on the $(N+1)$-fold tensor product $V_0 \otimes \left(\otimes_{k=1}^{N} V_k\right)$, by using the R-matrix $R^{VV}(\beta) \in \mathrm{End}(V \otimes V)$.

$$\mathcal{T}(\beta)=R^{VV}_{0,1}(\beta)\cdots R^{VV}_{0,N}(\beta)=\begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n-1} & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n-1} & A_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_{n-1,1} & A_{n-1,2} & \cdots & A_{n-1,n-1} & A_{n-1,n} \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n-1} & A_{n,n} \end{pmatrix} \tag{2.8}$$

where the partition into $n \times n$ blocks is according to the basis of V_0.

Let us set the transfer matrix $T(\beta)$ acting on the Nth-fold tensor product $\otimes_{k=1}^{N} V_k$ by taking the trace of the monodromy matrix $\mathcal{T}(\beta) \in \mathrm{End}(V_0 \otimes (\otimes_{k=1}^{N} V_k))$,

$$T(\beta)=\sum_{j=1}^{n} A_{j,j}=\mathrm{tr}_{V_0}(\mathcal{T}(\beta)). \tag{2.9}$$

From the Yang–Baxter equation (2.6), we know the transfer matrices $T(\beta) \in \mathrm{End}(\otimes_{k=1}^{N} V_k)$ commute each other,

$$[T(\beta_1), T(\beta_2)]=0. \tag{2.10}$$

In the thermodynamic limit $N \to \infty$, the logarithmic derivative of the transfer matrix $T(\beta)$ and the Hamiltonian $\mathcal{H}$ (1.1) have the following relation:

$$\left(\frac{\mathrm{d}}{\mathrm{d}\beta}\log T\right)(0) \sim \mathcal{H}. \tag{2.11}$$

In this way the Abelian symmetry emerges from the Yang–Baxter equation.

2.3. *S-matrix*

The purpose of this subsection is to present the S-matrix (scattering matrix) of this model, and to explain the relation between this model and the $A^{(1)}_{n-1}$ affine Toda fields with imaginary coupling [11].

Doikou and Nepomechie [13] computed the scattering matrix for the critical $A^{(1)}_{n-1}$ spin chain, by means of the Bethe ansatz. The S-matrix $S(\beta) \in \mathrm{End}(V \otimes V)$ of the present model is given by

$$\begin{aligned} S(\beta) &= s(\beta)\bar{S}(\beta) \\ s(\beta) &= \frac{S_2(-\mathrm{i}\beta|\frac{2\pi}{n}(\xi-1),2\pi)S_2(\mathrm{i}\beta+\frac{2(n-1)\pi}{n}|\frac{2\pi}{n}(\xi-1),2\pi)}{S_2(\mathrm{i}\beta|\frac{2\pi}{n}(\xi-1),2\pi)S_2(-\mathrm{i}\beta+\frac{2(n-1)\pi}{n}|\frac{2\pi}{n}(\xi-1),2\pi)}. \end{aligned} \tag{2.12}$$

The auxiliary operator $\bar{S}(\beta) \in \mathrm{End}(V \otimes V)$ is given as follows:

$$\bar{S}(\beta) v_{k_1} \otimes v_{k_2} = \sum_{j_1, j_2=0}^{n-1} v_{j_1} \otimes v_{j_2} \bar{S}(\beta)_{j_1 j_2}^{k_1 k_2}. \tag{2.13}$$

Here the non-zero entries of the matrix elements $\bar{S}(\beta)_{j_1 j_2}^{k_1 k_2}$ are

$$\bar{S}(\beta)_{jj}^{jj} = 1 \tag{2.14}$$

$$\bar{S}(\beta)_{jk}^{jk} = -\frac{\mathrm{sh}\left(\frac{n}{2(\xi-1)}\beta\right)}{\mathrm{sh}\left(\frac{n}{2(\xi-1)}(\beta - \frac{2\pi \mathrm{i}}{n})\right)} \qquad (j \neq k) \tag{2.15}$$

$$\bar{S}(\beta)_{jk}^{kj} = \begin{cases} -\dfrac{\mathrm{e}^{-\frac{n}{2(\xi-1)}\beta}\,\mathrm{sh}\left(\frac{\pi \mathrm{i}}{(\xi-1)}\right)}{\mathrm{sh}\left(\frac{n}{2(\xi-1)}(\beta - \frac{2\pi \mathrm{i}}{n})\right)} & (j < k) \\ -\dfrac{\mathrm{e}^{\frac{n}{2(\xi-1)}\beta}\,\mathrm{sh}\left(\frac{\pi \mathrm{i}}{(\xi-1)}\right)}{\mathrm{sh}\left(\frac{n}{2(\xi-1)}(\beta - \frac{2\pi \mathrm{i}}{n})\right)} & (j > k). \end{cases} \tag{2.16}$$

The scattering matrix of the present model agrees with those of the $A_{n-1}^{(1)}$ affine Toda fields theory with imaginary coupling [11]. When we assume that the physical space of both models is the same space, the type II vertex operators of the critical $A_{n-1}^{(1)}$ chain coincide with the Zamolodchikov–Faddeev operators of the $A_{n-1}^{(1)}$ affine Toda fields with imaginary coupling [11]. If the reader is not familiar with the terminology 'type I, type II vertex operators', see the textbook [1].

3. Vertex operators

The purpose of this section is to present the commutation relations of the type I vertex operators $\Phi_j(\beta)$ and the dual vertex operators $\Phi_j^*(\beta)$.

The type I vertex $\Phi_j(\beta)$ operators satisfy

$$\Phi_{j_2}(\beta_1)\Phi_{j_1}(\beta_2) = \sum_{k_1 k_2=0}^{n-1} R^{VV}(\beta_1 - \beta_2)_{j_1 j_2}^{k_1 k_2} \Phi_{k_1}(\beta_2)\Phi_{k_2}(\beta_1). \tag{3.1}$$

Here $R^{VV}(\beta)_{j_1 j_2}^{k_1 k_2}$ are matrix elements of the R-matrix $R^{VV}(\beta) \in \mathrm{End}(V \otimes V)$,

$$R^{VV}(\beta) v_{k_1} \otimes v_{k_2} = \sum_{j_1, j_2=0}^{n-1} v_{j_1} \otimes v_{j_2} R^{VV}(\beta)_{j_1 j_2}^{k_1 k_2}. \tag{3.2}$$

The dual type I vertex $\Phi_j^*(\beta)$ operators satisfy

$$\Phi_{j_2}^*(\beta_1)\Phi_{j_1}^*(\beta_2) = \sum_{k_1 k_2=0}^{n-1} R^{V^*V^*}(\beta_1 - \beta_2)_{j_1 j_2}^{k_1 k_2} \Phi_{k_1}^*(\beta_2)\Phi_{k_2}^*(\beta_1) \tag{3.3}$$

where we have set $R^{V^*V^*}(\beta)_{j_1 j_2}^{k_1 k_2} = R^{VV}(\beta)_{j_2 j_1}^{k_2 k_1}$.

Let us set $R^{V^*V^*}(\beta) \in \mathrm{End}(V^* \otimes V^*)$ by

$$R^{V^*V^*}(\beta) v_{k_1}^* \otimes v_{k_2}^* = \sum_{j_1, j_2=0}^{n-1} v_{j_1}^* \otimes v_{j_2}^* R^{V^*V^*}(\beta)_{j_1 j_2}^{k_1 k_2}. \tag{3.4}$$

The type I vertex operators $\Phi_j(\beta)$ and dual type I vertex operators $\Phi_j^*(\beta)$ satisfy

$$\Phi_{j_2}(\beta_1)\Phi_{j_1}^*(\beta_2) = \sum_{k_1k_2=0}^{n-1} R^{V^*V}(\beta_1-\beta_2)_{j_1j_2}^{k_1k_2}\Phi_{k_1}^*(\beta_2)\Phi_{k_2}(\beta_1) \tag{3.5}$$

where we have set $R^{V^*V}(\beta)_{j_1j_2}^{k_1k_2}$ as matrix elements of the operator $R^{V^*V}(\beta)\in \mathrm{End}(V^*\otimes V)$. Here we have set $R^{V^*V}(\beta)\in \mathrm{End}(V^*\otimes V)$ by

$$R^{V^*V}(\beta) = r^*(\beta)\bar{R}^*(\beta) \qquad r^*(\beta) = \frac{S_2(-\mathrm{i}\alpha+\pi|\frac{2\pi}{n}\xi,2\pi)S_2(\mathrm{i}\alpha+\pi+\frac{2\pi}{n}|\frac{2\pi}{n}\xi,2\pi)}{S_2(\mathrm{i}\alpha+\pi|\frac{2\pi}{n}\xi,2\pi)S_2(-\mathrm{i}\alpha+\pi+\frac{2\pi}{n}|\frac{2\pi}{n}\xi,2\pi)} \tag{3.6}$$

where the auxiliary operator $\bar{R}^*(\beta)$ is given as follows:

$$\bar{R}^*(\beta)v_{k_1}^*\otimes v_{k_2} = \sum_{j_1,j_2=0}^{n-1} v_{j_1}^*\otimes v_{j_2}\bar{R}^*(\beta)_{j_1j_2}^{k_1k_2} \tag{3.7}$$

where the non-zero entries are

$$\bar{R}^*(\beta)_{jk}^{jk} = 1 \qquad (j\neq k) \tag{3.8}$$

$$\bar{R}^*(\beta)_{jj}^{jj} = -\frac{\mathrm{sh}\left(\frac{n}{2\xi}\beta\right)}{\mathrm{sh}\left(\frac{n}{2\xi}(\beta+\frac{2\pi\mathrm{i}}{n})\right)} \tag{3.9}$$

$$\bar{R}^*(\beta)_{jj}^{kk} = \begin{cases} \dfrac{\mathrm{e}^{-\frac{n}{2\xi}\beta}\,\mathrm{sh}\left(\frac{\pi}{\xi}\mathrm{i}\right)}{\mathrm{sh}\left(\frac{n}{2\xi}(\beta+\frac{2\pi\mathrm{i}}{n})\right)} & (j>k) \\ \dfrac{\mathrm{e}^{\frac{n}{2\xi}\beta}\,\mathrm{sh}\left(\frac{\pi}{\xi}\mathrm{i}\right)}{\mathrm{sh}\left(\frac{n}{2\xi}(\beta+\frac{2\pi\mathrm{i}}{n})\right)} & (j<k). \end{cases} \tag{3.10}$$

The type I vertex operators $\Phi_j(\beta)$ and the dual type I vertex operators $\Phi_j^*(\beta)$ satisfy the inversion relation.

$$\Phi_{j_1}(\beta)\Phi_{j_2}^*(\beta+\pi\mathrm{i}) = \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}j_1}\delta_{j_1,j_2}\mathrm{id} \qquad (j_1\leqslant j_2). \tag{3.11}$$

4. Free boson realizations

The purpose of this section is to present the free boson realizations of the vertex operators.

Let us introduce Bose fields $b_j(t)$, $(j\in\{0,1,\ldots,n-1\}, t\in\mathbb{R})$ as

$$[b_j(t),b_k(t')] = -\frac{1}{t}\frac{\mathrm{sh}\left(\frac{\pi}{n}(a_j|a_k)t\right)\mathrm{sh}\left(\frac{\pi}{n}(\xi-1)t\right)}{\mathrm{sh}\left(\frac{\pi}{n}t\right)\mathrm{sh}\left(\frac{\pi}{n}\xi t\right)}\delta(t+t'). \tag{4.1}$$

Let us introduce the Fock space $\mathcal{F}$ generated by the vacuum vector $|vac\rangle$ which satisfies

$$b(t)|vac\rangle = 0 \qquad \text{if} \quad t>0. \tag{4.2}$$

Let us set the auxiliary operators $b_1^*(t)$ and $b_{n-1}^*(t)$ by

$$b_1^*(t) = -\sum_{j=1}^{n-1} b_j(t)\frac{\mathrm{sh}\,\frac{(n-j)\pi t}{n}}{\mathrm{sh}\,\pi t} \tag{4.3}$$

$$b_{n-1}^*(t) = -\sum_{j=1}^{n-1} b_j(t)\frac{\mathrm{sh}\,\frac{j\pi t}{n}}{\mathrm{sh}\,\pi t}. \tag{4.4}$$

We have the following commutation relation:

$$[b_1^*(t), b_j(t')] = \delta_{j1}\frac{1}{t}\frac{\mathrm{sh}\left(\frac{\pi}{n}(\xi-1)t\right)}{\mathrm{sh}\left(\frac{\pi}{n}\xi t\right)}\delta(t+t') \tag{4.5}$$

$$[b_j(t), b_{n-1}^*(t')] = \delta_{j,n-1}\frac{1}{t}\frac{\mathrm{sh}\left(\frac{\pi}{n}(\xi-1)t\right)}{\mathrm{sh}\left(\frac{\pi}{n}\xi t\right)}\delta(t+t'). \tag{4.6}$$

Let us set the basic operators as

$$U_j(\beta) = \,:\exp\left(-\int_{-\infty}^{\infty} b_j(t)\,\mathrm{e}^{\mathrm{i}\beta t}\,\mathrm{d}t\right): \qquad (1 \leqslant j \leqslant n-1) \tag{4.7}$$

$$U_0(\beta) = \,:\exp\left(-\int_{-\infty}^{\infty} b_1^*(t)\,\mathrm{e}^{\mathrm{i}\beta t}\,\mathrm{d}t\right): \tag{4.8}$$

$$U_n(\beta) = \,:\exp\left(-\int_{-\infty}^{\infty} b_{n-1}^*(t)\,\mathrm{e}^{\mathrm{i}\beta t}\,\mathrm{d}t\right):. \tag{4.9}$$

The bosonization of the type I vertex operators are given by

$$\begin{aligned}\Phi_j(\beta) &= \int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_1}{2\pi\mathrm{i}}\cdots\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_j}{2\pi\mathrm{i}}\frac{\mathrm{e}^{\frac{n}{2\xi}(\alpha_j-\beta)}U_0(\beta)U_1(\alpha_1)\cdots U_j(\alpha_j)}{\prod_{k=1}^{j}\mathrm{sh}\left(\frac{n}{2\xi}(\alpha_{k-1}-\alpha_k-\frac{\pi\mathrm{i}}{n})\right)}\\ &= \int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_1}{2\pi\mathrm{i}}\cdots\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_j}{2\pi\mathrm{i}}\frac{\mathrm{e}^{\frac{n}{2\xi}(\alpha_j-\beta)}U_j(\alpha_j)U_{j-1}(\alpha_{j-1})\cdots U_0(\beta)}{\prod_{k=1}^{j}\mathrm{sh}\left(\frac{n}{2\xi}(\alpha_k-\alpha_{k-1}-\frac{\pi\mathrm{i}}{n})\right)}\\ &\quad (0\leqslant j\leqslant n-1)\end{aligned} \tag{4.10}$$

where $\alpha_0 = \beta$.

The bosonization of the dual type I vertex operators are given by

$$\begin{aligned}\Phi_j^*(\beta) &= g_n\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_{j+1}}{2\pi\mathrm{i}}\cdots\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_{n-1}}{2\pi\mathrm{i}}\frac{\mathrm{e}^{\frac{n}{2\xi}(\alpha_{j+1}-\beta)}U_{j+1}(\alpha_{j+1})\cdots U_{n-1}(\alpha_{n-1})U_n(\beta)}{\prod_{k=j+1}^{n-1}\mathrm{sh}\left(\frac{n}{2\xi}(\alpha_k-\alpha_{k+1}-\frac{\pi\mathrm{i}}{n})\right)}\\ &= g_n\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_{j+1}}{2\pi\mathrm{i}}\cdots\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_{n-1}}{2\pi\mathrm{i}}\frac{\mathrm{e}^{\frac{n}{2\xi}(\alpha_{j+1}-\beta)}U_n(\beta)U_{n-1}(\alpha_{n-1})\cdots U_{j+1}(\alpha_{j+1})}{\prod_{k=j+1}^{n-1}\mathrm{sh}\left(\frac{n}{2\xi}(\alpha_{k+1}-\alpha_k-\frac{\pi\mathrm{i}}{n})\right)}\\ &\quad (0\leqslant j\leqslant n-1)\end{aligned} \tag{4.11}$$

where $\alpha_n = \beta$. Here we set the constant g_n by

$$g_n = \left(\frac{n}{2\xi}\right)^{n-1}\mathrm{e}^{\frac{\xi-1}{\xi}(\gamma+\log\frac{2\pi\xi}{n})n+\frac{\pi\mathrm{i}}{n}(n-1)}\frac{\Gamma(1-1/\xi)}{\mathrm{sh}(\pi\mathrm{i}/\xi)\Gamma(1/\xi)^{n-1}}. \tag{4.12}$$

Miwa and Takeyama [11] gave the free boson realization of some vertex operators, to construct the solutions of the critical quantum Knizhnik–Zamolodchikov equations at level 0. The calculation of the S-matrix of the critical $A_{n-1}^{(1)}$ chain [13] teaches us that their vertex operators are the type II vertex operators of the present model (see section 2.3).

5. Proof

The purpose of this section is to show that our bosonizations of the vertex operators (4.10) and (4.11) satisfy the commutation relations (2.1), (3.4), (3.6), and the inversion relation (3.11).

We prove those relations by using contraction formulae and the commutation relations of the basic operators $U_j(\beta)$, summarized in appendix B. We give proofs of the vertex operators along the line of [15, 16].

First, we introduce the notion of 'weak equality'. Consider an integral of the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\, U_j(\alpha_1)U_j(\alpha_2)F(\alpha_1,\alpha_2). \tag{5.1}$$

Due to the commutation relation of $U_j(\alpha)$, the above integral equals

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}\alpha_1\,\mathrm{d}\alpha_2\, U_j(\alpha_1)U_j(\alpha_2)F(\alpha_2,\alpha_1)H_{j,j}(\alpha_2-\alpha_1). \tag{5.2}$$

Observing this we define 'weak equality' in the following sense. We say that the functions $G_1(\alpha_1,\alpha_2)$ and $G_2(\alpha_1,\alpha_2)$ are equal in a weak sense if

$$G_1(\alpha_1,\alpha_2)+H_{j,j}(\alpha_2-\alpha_1)G_1(\alpha_2,\alpha_1) = G_2(\alpha_1,\alpha_2)+H_{j,j}(\alpha_2-\alpha_1)G_2(\alpha_2,\alpha_1). \tag{5.3}$$

We write 'weak equality' of the integrand as

$$G_1(\alpha_1,\alpha_2)\sim G_2(\alpha_1,\alpha_2). \tag{5.4}$$

Let us start a proof of the commutation relations of the vertex operators.

Proof of (3.1). We want to prove the relations

$$\Phi_\mu(\alpha_0)\Phi_\mu(\alpha_0') = r(\alpha_0-\alpha_0')\Phi_\mu(\alpha_0')\Phi_\mu(\alpha_0) \qquad (0\leqslant\mu\leqslant n-1) \tag{5.5}$$

$$\begin{aligned}\Phi_\mu(\alpha_0)\Phi_\nu(\alpha_0') = r(\alpha_0-\alpha_0')\big(&b(\alpha_0-\alpha_0')\Phi_\nu(\alpha_0')\Phi_\mu(\alpha_0)\\ &+\mathrm{e}^{\mathrm{sgn}(\nu-\mu)\rho(\alpha_0-\alpha_0')}c(\alpha_0-\alpha_0')\Phi_\mu(\alpha_0')\Phi_\nu(\alpha_0)\big).\end{aligned} \tag{5.6}$$

In order to prove the above relations it is enough to prove the equalities of the integrand parts in a weak sense. First, we will show the equality (5.5). In what follows we use the following abbreviations:

$$b(\alpha) = -\frac{\mathrm{sh}\,\rho\alpha}{\mathrm{sh}\,\rho(\alpha+\frac{2\pi\mathrm{i}}{n})} \qquad c(\alpha) = \frac{\mathrm{sh}\,\rho\frac{2\pi\mathrm{i}}{n}}{\mathrm{sh}\,\rho(\alpha+\frac{2\pi\mathrm{i}}{n})} \qquad d(\alpha) = \frac{\mathrm{e}^{\rho\alpha}}{\mathrm{sh}\,\rho(-\alpha-\frac{\pi\mathrm{i}}{n})} \tag{5.7}$$

$$I(\alpha) = H_{j-1,j}(\alpha) \qquad \rho = \frac{n}{2\xi}. \tag{5.8}$$

For the $\mu=0$ case of (5.5), it is just the commutation relation of $U_0(\alpha)$ (see appendix B). For the $\mu\geqslant 1$ case we will show that by induction. By using the commutation relations of the basic operators, we can rearrange the operator part of (5.5) as

$$U_0(\alpha_0)U_0(\alpha_0')U_1(\alpha_1)U_1(\alpha_1')\cdots U_\mu(\alpha_\mu)U_\mu(\alpha_\mu'). \tag{5.9}$$

The integrand part of the left-hand side of (5.5), (we use the notation L_μ) becomes

$$L_\mu(\alpha_0\alpha_0'\cdots\alpha_\mu\alpha_\mu') = \prod_{k=1}^{\mu} d(\alpha_k-\alpha_{k-1})\prod_{k=1}^{\mu} d(\alpha_k'-\alpha_{k-1}')\prod_{k=1}^{\mu} I(\alpha_k-\alpha_{k-1}'). \tag{5.10}$$

The integrand of the right-hand side (we use the notation R_μ) is given by the exchange of variables $\alpha_0\leftrightarrow\alpha_0'$ of the left-hand side,

$$R_\mu(\alpha_0\alpha_0'\cdots\alpha_\mu\alpha_\mu') = L_\mu(\alpha_0'\alpha_0\cdots\alpha_\mu\alpha_\mu'). \tag{5.11}$$

For the $\mu=1$ case the following weak identity for variables α_1 and α_1' can be shown by an exact calculation:

$$L_1(\alpha_0\alpha_0'\alpha_1\alpha_1')\sim R_1(\alpha_0\alpha_0'\alpha_1\alpha_1'). \tag{5.12}$$

Because the integrand functions L_μ and R_μ split into two parts, i.e.

$$L_\mu(\alpha_0\alpha_0'\alpha_1\alpha_1'\cdots\alpha_\mu\alpha_\mu') = L_1(\alpha_0\alpha_0'\alpha_1\alpha_1')L_{\mu-1}(\alpha_1\alpha_1'\alpha_2\alpha_2'\cdots\alpha_\mu\alpha_\mu') \tag{5.13}$$

the case $\mu \geqslant 2$ follows by induction,

$$L_\mu(\alpha_0\alpha_0'\alpha_1\alpha_1'\cdots\alpha_\mu\alpha_\mu') \sim R_\mu(\alpha_0\alpha_0'\alpha_1\alpha_1'\cdots\alpha_\mu\alpha_\mu'). \tag{5.14}$$

We have proved the equality (5.5). □

Next we will show

$$\Phi_\mu(\alpha_0)\Phi_\nu(\alpha_0') = r(\alpha_0-\alpha_0')\big(b(\alpha_0-\alpha_0')\Phi_\nu(\alpha_0')\Phi_\mu(\alpha_0) \\ + \mathrm{e}^{\mathrm{sgn}(\nu-\mu)\rho(\alpha_0-\alpha_0')}c(\alpha_0-\alpha_0')\Phi_\mu(\alpha_0')\Phi_\nu(\alpha_0)\big). \tag{5.15}$$

Proof. We prove the case $\nu > \mu$. The case $\mu > \nu$ is similar. For the case $\nu > \mu = 0$, the equality (5.15) follows from the following integrand equality, which can be derived by direct calculations.

$$d(\alpha_1'-\alpha_0') = b(\alpha_0-\alpha_0')I(\alpha_1'-\alpha_0)d(\alpha_1'-\alpha_0') + c(\alpha_0-\alpha_0')\mathrm{e}^{\rho(\alpha_0-\alpha_0')}d(\alpha_1'-\alpha_0). \tag{5.16}$$

For the case $\nu > \mu \geqslant 1$, the equality (5.15) follows from the weak equality with respect to the variables $(\alpha_1\alpha_1'), (\alpha_2\alpha_2')\cdots(\alpha_\mu\alpha_\mu')$:

$$A_\mu(\alpha_0\alpha_0'\cdots\alpha_{\mu+1}\alpha_{\mu+1}') + B_\mu(\alpha_0\alpha_0'\cdots\alpha_{\mu+1}\alpha_{\mu+1}') + C_\mu(\alpha_0\alpha_0'\cdots\alpha_{\mu+1}\alpha_{\mu+1}') \sim 0 \tag{5.17}$$

where we set

$$A_\mu(\alpha_0\alpha_0'\cdots\alpha_{\mu+1}\alpha_{\mu+1}') = \prod_{k=0}^{\mu-1} I(\alpha_{k+1}-\alpha_k')\prod_{k=0}^{\mu-1} d(\alpha_{k+1}-\alpha_k)\prod_{k=0}^{\mu} d(\alpha_{k+1}'-\alpha_k') \tag{5.18}$$

$$B_\mu(\alpha_0\alpha_0'\cdots\alpha_{\mu+1}\alpha_{\mu+1}') = -b(\alpha_0-\alpha_0')I(\alpha_{\mu+1}'-\alpha_\mu')\left\{\prod_{k=1}^{\mu-1} d(\alpha_{k+1}-\alpha_k')\right\}I(\alpha_1-\alpha_0) \\ \times d(\alpha_{\mu+1}'-\alpha_\mu)\left\{\prod_{k=1}^{\mu-1} d(\alpha_{k+1}-\alpha_k)\right\}d(\alpha_1-\alpha_0') \\ \times\left\{\prod_{k=1}^{\mu-1} d(\alpha_{k+1}'-\alpha_k')\right\}d(\alpha_1'-\alpha_0) \tag{5.19}$$

and

$$C_\mu(\alpha_0\alpha_0'\cdots\alpha_{\mu+1}\alpha_{\mu+1}') = -c(\alpha_0-\alpha_0')\mathrm{e}^{\rho(\alpha_0-\alpha_0')}\left\{\prod_{k=1}^{\mu-1} I(\alpha_{k+1}-\alpha_k')\right\}I(\alpha_1-\alpha_0) \\ \times\left\{\prod_{k=1}^{\mu} d(\alpha_{k+1}'-\alpha_k')\right\}d(\alpha_1'-\alpha_0)\left\{\prod_{k=1}^{\mu} d(\alpha_{k+1}-\alpha_k)\right\}d(\alpha_1-\alpha_0'). \tag{5.20}$$

We want to prove equation (5.17) by induction of μ. Inserting equation (5.17) for $\mu-1$ into B_μ, we have the equation in a weak sense with respect to the variables $(\alpha_1\alpha_1')\cdots(\alpha_\mu\alpha_\mu')$:

$$B_\mu(\alpha_0\alpha_0'\cdots\alpha_{\mu+1}\alpha_{\mu+1}') \sim A_\mu'(\alpha_0\alpha_0'\cdots\alpha_{\mu+1}\alpha_{\mu+1}') + C_\mu'(\alpha_0\alpha_0'\cdots\alpha_{\mu+1}\alpha_{\mu+1}') \tag{5.21}$$

where we set

$$A'_\mu(\alpha_0\alpha'_0\cdots\alpha_\mu\alpha'_{\mu+1}) = -\frac{b(\alpha_0-\alpha'_0)}{b(\alpha'_1-\alpha_1)}\left\{\prod_{k=2}^{\mu} d(\alpha'_{k+1}-\alpha'_k)\right\} d(\alpha'_2-\alpha_1)d(\alpha_1-\alpha'_0)$$
$$\times\left\{\prod_{k=2}^{\mu-1} d(\alpha_{k+1}-\alpha_k)\right\} d(\alpha_2-\alpha'_1)d(\alpha'_1-\alpha_0)$$
$$\times\left\{\prod_{k=2}^{\mu-1} I(\alpha_{k+1}-\alpha'_k)\right\} I(\alpha_2-\alpha_1)I(\alpha_1-\alpha_0) \tag{5.22}$$

and

$$C'_\mu(\alpha_0\alpha'_0\cdots\alpha_{\mu+1}\alpha'_{\mu+1}) = \frac{b(\alpha_0-\alpha'_0)c(\alpha'_1-\alpha_1)\mathrm{e}^{\rho(\alpha'_1-\alpha_1)}}{b(\alpha'_1-\alpha_1)}\left\{\prod_{k=1}^{\mu} d(\alpha'_{k+1}-\alpha'_k)\right\} d(\alpha'_1-\alpha_0)$$
$$\times\left\{\prod_{k=1}^{\mu-1} d(\alpha_{k+1}-\alpha_k)\right\} d(\alpha_1-\alpha'_0)\left\{\prod_{k=1}^{\mu-1} I(\alpha_{k+1}-\alpha'_k)\right\} I(\alpha_1-\alpha_0). \tag{5.23}$$

Using the relation $b(\alpha)H(\alpha)=b(-\alpha)$, we have

$$A'_\mu(\alpha_0\alpha'_0\cdots\alpha_{\mu+1}\alpha'_{\mu+1}) \sim A''_\mu(\alpha_0\alpha'_0\cdots\alpha_{\mu+1}\alpha'_{\mu+1}) \tag{5.24}$$

where

$$A''_\mu(\alpha_0\alpha'_0\cdots\alpha_{\mu+1}\alpha'_{\mu+1}) = -\frac{b(\alpha_0-\alpha'_0)}{b(\alpha'_1-\alpha_1)}\left\{\prod_{k=0}^{\mu} d(\alpha'_{k+1}-\alpha'_k)\right\}\left\{\prod_{k=0}^{\mu} d(\alpha_{k+1}-\alpha_k)\right\}$$
$$\times\left\{\prod_{k=0}^{\mu-1} I(\alpha_{k+1}-\alpha'_k)\right\} I(\alpha'_1-\alpha_0). \tag{5.25}$$

We have

$$A_\mu + A''_\mu \sim \left\{\mathrm{sh}\left(\rho\left(\alpha'_0-\alpha'_1-\frac{\pi\mathrm{i}}{n}\right)\right)\mathrm{sh}\left(\rho\left(\alpha_0-\alpha_1+\frac{\pi\mathrm{i}}{n}\right)\right)\mathrm{sh}\left(\rho\left(\alpha_0+\alpha_1-\alpha'_0-\alpha'_1\right)\right)\right.$$
$$\left.\times\,\mathrm{sh}\left(\rho\left(\frac{2\pi\mathrm{i}}{n}\right)\right)\right\}\left\{\mathrm{sh}\left(\rho\left(\alpha'_0-\alpha_1-\frac{\pi\mathrm{i}}{n}\right)\right)\mathrm{sh}\left(\rho\left(\alpha'_1-\alpha_0+\frac{2\pi\mathrm{i}}{n}\right)\right)\right.$$
$$\left.\times\,\mathrm{sh}\left(\rho\left(\alpha_0-\alpha'_1-\frac{\pi\mathrm{i}}{n}\right)\right)\mathrm{sh}\left(\rho\left(\alpha_0-\alpha'_0+\frac{2\pi\mathrm{i}}{n}\right)\right)\right\}^{-1}$$
$$\times\frac{1}{b(\alpha'_1-\alpha_1)}\left\{\prod_{k=0}^{\mu} d(\alpha'_{k+1}-\alpha'_k)\right\}\left\{\prod_{k=0}^{\mu-1} d(\alpha_{k+1}-\alpha_k)\right\}$$
$$\times\left\{\prod_{k=1}^{\mu-1} I(\alpha_{k+1}-\alpha'_k)\right\} \tag{5.26}$$

where we have used the relation

$$I(\alpha_1-\alpha'_0)b(\alpha'_1-\alpha_1) - I(\alpha'_1-\alpha_0)b(\alpha_0-\alpha'_0) = \left\{\mathrm{sh}\left(\rho\left(\alpha'_0-\alpha'_1-\frac{\pi\mathrm{i}}{n}\right)\right)\right.$$
$$\left.\times\,\mathrm{sh}\left(\rho\left(\alpha_0-\alpha_1+\frac{\pi\mathrm{i}}{n}\right)\right)\mathrm{sh}\left(\rho\left(\alpha_0+\alpha_1-\alpha'_0-\alpha'_1\right)\right)\mathrm{sh}\left(\rho\left(\frac{2\pi\mathrm{i}}{n}\right)\right)\right\}$$

$$\times\left\{\mathrm{sh}\left(\rho\left(\alpha_0'-\alpha_1-\frac{\pi\mathrm{i}}{n}\right)\right)\mathrm{sh}\left(\rho\left(\alpha_1'-\alpha_0+\frac{2\pi\mathrm{i}}{n}\right)\right)\right.$$

$$\left.\times\,\mathrm{sh}\left(\rho\left(\alpha_0-\alpha_1'-\frac{\pi\mathrm{i}}{n}\right)\right)\mathrm{sh}\left(\rho(\alpha_0-\alpha_0'+\frac{2\pi\mathrm{i}}{n})\right)\right\}^{-1}. \tag{5.27}$$

We have

$$C_\mu+C_\mu'=-b(\alpha_0-\alpha_0')\frac{\mathrm{sh}\left(\rho(\frac{2\pi\mathrm{i}}{n})\right)\mathrm{sh}\left(\rho(\alpha_0+\alpha_1-\alpha_0'-\alpha_1')\right))}{\mathrm{sh}\left(\rho(\alpha_1'-\alpha_1)\right)\mathrm{sh}\left(\rho(\alpha_0-\alpha_0')\right))}$$

$$\times\left\{\prod_{k=1}^{\mu}d(\alpha_{k+1}'-\alpha_k')\right\}d(\alpha_1'-\alpha_0)\left\{\prod_{k=1}^{\mu-1}d(\alpha_{k+1}-\alpha_k)\right\}d(\alpha_1-\alpha_0')$$

$$\times\left\{\prod_{k=1}^{\mu-1}I(\alpha_{k+1}-\alpha_k')\right\}I(\alpha_1-\alpha_0) \tag{5.28}$$

where we have used the relation

$$\frac{c(\alpha_1'-\alpha_1)}{b(\alpha_1'-\alpha_1)}\mathrm{e}^{\rho(\alpha_1'-\alpha_1)}-\frac{c(\alpha_0-\alpha_0')}{b(\alpha_0-\alpha_0')}\mathrm{e}^{\rho(\alpha_0-\alpha_0')}=-\frac{\mathrm{sh}\left(\rho(\frac{2\pi\mathrm{i}}{n})\right)\mathrm{sh}\left(\rho(\alpha_0+\alpha_1-\alpha_0'-\alpha_1')\right))}{\mathrm{sh}\left(\rho(\alpha_1'-\alpha_1)\right)\mathrm{sh}\left(\rho(\alpha_0-\alpha_0')\right))}. \tag{5.29}$$

We arrive at

$$A_\mu+B_\mu+C_\mu\sim A_\mu+A_\mu''+C_\mu+C_\mu'\sim 0. \tag{5.30}$$

Now we have proved the commutation relation (2.1). □

Using the same arguments as above we can prove the commutation relations (3.3) and (3.5).

Next we prove the inversion relations.

Proof of (3.11). Using the contraction relations of the basic operators $U_j(\beta)$ in appendix B, we give direct calculations of the inversion relation (3.11).

The free boson realizations of the type I vertex operator are deformed as

$$\Phi_j(\beta)=\left(\frac{\mathrm{i}}{\pi}\right)^j\mathrm{e}^{-j\frac{\xi-1}{\xi}(\gamma+\log\frac{2\pi\xi}{n})}\int_{C_j}\frac{\mathrm{d}\alpha_j}{2\pi\mathrm{i}}\cdots\int_{C_1}\frac{\mathrm{d}\alpha_1}{2\pi\mathrm{i}}\mathrm{e}^{\frac{n}{2\xi}(\alpha_j-\beta)}:U_0(\beta)\cdots U_j(\alpha_j):$$

$$\times\prod_{k=1}^{j}\Gamma\left(\frac{n}{2\pi\xi}\mathrm{i}(\alpha_k-\alpha_{k-1})+\frac{1}{2\xi}\right)\Gamma\left(-\frac{n}{2\pi\xi}\mathrm{i}(\alpha_k-\alpha_{k-1})+\frac{1}{2\xi}\right) \tag{5.31}$$

where $\alpha_0=\beta$. Here the contour C_k $(k=1,\ldots,j)$ is taken as $(-\infty,\infty)$ except that the poles

$$\alpha_k-\alpha_{k-1}=\frac{\pi\mathrm{i}}{n}+\frac{2\pi\xi i}{n}l\qquad(l\in\mathbb{N}\geqslant 0) \tag{5.32}$$

of $\Gamma\left(\frac{n}{2\pi\xi}\mathrm{i}(\alpha_k-\alpha_{k-1})+\frac{1}{2\xi}\right)$ are above C_k and the poles

$$\alpha_k-\alpha_{k-1}=-\frac{\pi\mathrm{i}}{n}-\frac{2\pi\xi i}{n}l\qquad(l\in\mathbb{N}\geqslant 0) \tag{5.33}$$

of $\Gamma\left(-\frac{n}{2\pi\xi}\mathrm{i}(\alpha_k-\alpha_{k-1})+\frac{1}{2\xi}\right)$ are below C_k.

The free boson realizations of the dual type I vertex operator are deformed as

$$\Phi^*_j(\beta) = g_n \left(\frac{\mathrm{i}}{\pi}\right)^{n-j-1} \mathrm{e}^{-(n-j-1)\frac{\xi-1}{\xi}(\gamma+\log\frac{2\pi\xi}{n})} \int_{C^*_{j+1}} \frac{\mathrm{d}\alpha_{j+1}}{2\pi\mathrm{i}} \cdots \int_{C^*_{n-1}} \frac{\mathrm{d}\alpha_{n-1}}{2\pi\mathrm{i}} \mathrm{e}^{\frac{n}{2\xi}(\alpha_{j+1}-\beta)}$$
$$\times : U_{j+1}(\alpha_{j+1}) \cdots U_n(\beta) : \prod_{k=j+1}^{n-1} \Gamma\left(\frac{n}{2\pi\xi}\mathrm{i}(\alpha_{k+1}-\alpha_k) + \frac{1}{2\xi}\right)$$
$$\times \Gamma\left(-\frac{n}{2\pi\xi}\mathrm{i}(\alpha_{k+1}-\alpha_k) + \frac{1}{2\xi}\right) \tag{5.34}$$

where $\alpha_n = \beta$. Here the contour C^*_k $(k = j+1, \ldots, n-1)$ is taken as $(-\infty, \infty)$ except that the poles

$$\alpha_k - \alpha_{k+1} = \frac{\pi\mathrm{i}}{n} + \frac{2\pi\xi i}{n} l \qquad (l \in \mathbb{N} \geqslant 0) \tag{5.35}$$

of $\Gamma\left(-\frac{n}{2\pi\xi}\mathrm{i}(\alpha_{k+1}-\alpha_k) + \frac{1}{2\xi}\right)$ are above C^*_k and the poles

$$\alpha_k - \alpha_{k+1} = -\frac{\pi\mathrm{i}}{n} - \frac{2\pi\xi i}{n} l \qquad (l \in \mathbb{N} \geqslant 0) \tag{5.36}$$

of $\Gamma\left(\frac{n}{2\pi\xi}\mathrm{i}(\alpha_{k+1}-\alpha_k) + \frac{1}{2\xi}\right)$ are below C^*_k.

Let us prove the relation (3.11). For $j_1 < j_2$ we have

$$\Phi_{j_1}(\beta_1)\Phi^*_{j_2}(\beta_2) = r^*(\beta_1-\beta_2)\Phi^*_{j_2}(\beta_2)\Phi_{j_1}(\beta_1). \tag{5.37}$$

As $\beta_2 \to \beta_1 + \pi\mathrm{i}$, $r^*(\beta_1-\beta_2)$ becomes zero.

Next, we prove the case $j_1 = j_2$. We have

$$\Phi_j(\beta_1)\Phi^*_j(\beta_2) = r^*(\beta_1-\beta_2) \int_{C_1} \frac{\mathrm{d}\alpha_1}{2\pi\mathrm{i}} \cdots \int_{C_j} \frac{\mathrm{d}\alpha_j}{2\pi\mathrm{i}} \int_{C^*_{j+1}} \frac{\mathrm{d}\alpha_{j+1}}{2\pi\mathrm{i}} \cdots \int_{C^*_{n-1}} \frac{\mathrm{d}\alpha_{n-1}}{2\pi\mathrm{i}}$$
$$\times U_n(\beta_2)U_{n-1}(\alpha_{n-1}) \cdots U_1(\alpha_1)U_0(\beta_1) g_n \mathrm{e}^{\frac{n}{2\xi}(\alpha_j+\alpha_{j+1}-\beta_1-\beta_2)} \tag{5.38}$$
$$\times \mathrm{sh}\left(\frac{n}{2\xi}\left(\alpha_j - \alpha_{j+1} - \frac{\pi\mathrm{i}}{n}\right)\right) \prod_{k=1}^{n} \frac{1}{\mathrm{sh}\left(\frac{n}{2\xi}(\alpha_k - \alpha_{k-1} - \frac{\pi\mathrm{i}}{n})\right)}.$$

When we take the limit $\beta_2 \to \beta_1 + \pi\mathrm{i}$, the contour is pinched, but the function $r^*(\beta_1-\beta_2)$ has a zero. Therefore, the limit is evaluated by successively taking the residues at $\alpha_k = \alpha_{k-1} + \frac{\pi\mathrm{i}}{n}$ for $k = 1, \ldots, j$, and $\alpha_k = \alpha_{k+1} - \frac{\pi\mathrm{i}}{n}$ for $k = j+1, \ldots, n-1$, successively. The following relation is useful:

$$U_n(\beta+\pi\mathrm{i})U_{n-1}\left(\beta + \frac{n-1}{n}\pi\mathrm{i}\right) \cdots U_1\left(\beta + \frac{\pi\mathrm{i}}{n}\right) U_0(\beta)$$
$$= \exp\left\{-\frac{\xi-1}{\xi}\left(\gamma + \log\frac{2\pi\xi}{n}\right)n\right\}\Gamma(1/\xi)^n. \tag{5.39}$$

Now we have proved the inversion relation (3.11). □

In this section we have demonstrated that the free boson realizations of the vertex operators (4.10) and (4.11) satisfy the characterizing relations of the vertex operators (3.1), (3.3), (3.5) and (3.11).

6. Correlation functions

The purpose of this section is to evaluate the trace of the vertex operators (6.5), which represents the correlation function. We perform calculations by using free boson realizations of the vertex operators and their duals (4.10) and (4.11).

6.1. Solutions of qKZ

In this subsection we introduce the trace of the vertex operators, which satisfies the qKZ equations (1.5) and (1.6). Let us define the degree operator D on the Fock space $\mathcal{F}$, introduced in section 4,

$$Db_j(-t)|vac\rangle = tb_j(-t)|vac\rangle \qquad (t > 0). \tag{6.1}$$

We have

$$\mathrm{e}^{\lambda D} b(t)\mathrm{e}^{-\lambda D} = \mathrm{e}^{-\lambda t} b(t). \tag{6.2}$$

Therefore, the degree operator D has the homogeneity condition,

$$\mathrm{e}^{\lambda D} U_j(\beta)\mathrm{e}^{-\lambda D} = U_j(\beta + \mathrm{i}\lambda). \tag{6.3}$$

The vertex operator and the degree operator enjoy the homogeneity property,

$$\mathrm{e}^{-\lambda D}\Phi_j(\beta)\mathrm{e}^{\lambda D} = \Phi_j(\beta + \mathrm{i}\lambda) \qquad \mathrm{e}^{-\lambda D}\Phi_j^*(\beta)\mathrm{e}^{\lambda D} = \Phi_j^*(\beta + \mathrm{i}\lambda). \tag{6.4}$$

Now let us consider the trace functions for $\lambda > 0$ defined by

$$G^{(N)}(\beta_1 \cdots \beta_N|\beta_{N+1} \cdots \beta_{2N})_{\epsilon_1\cdots\epsilon_{2N}} = \frac{\mathrm{tr}_{\mathcal{F}}\left(\mathrm{e}^{-\lambda D}\Phi^*_{\epsilon_1}(\beta_1)\cdots\Phi^*_{\epsilon_N}(\beta_N)\Phi_{\epsilon_{N+1}}(\beta_{N+1})\cdots\Phi_{\epsilon_{2N}}(\beta_{2N})\right)}{\mathrm{tr}_{\mathcal{F}}\left(\mathrm{e}^{-\lambda D}\right)} \tag{6.5}$$

where the space $\mathcal{F}$ is the Fock space of the free bosons.

By using the homogeneity condition (6.4) and the commutation relations (2.1), (3.4) and (3.6), it is shown that the above trace function satisfies the desired difference equations (1.5) and (1.6). The inversion relation (3.11) means the normalization condition (1.7).

6.2. Calculations of trace

Using the free boson realizations, we shall treat a trace of the form $\mathrm{tr}_{\mathcal{F}}(\mathrm{e}^{-\lambda D}\mathcal{O})$. The calculation is simplified by the technique of Clavelli and Shapiro [14]. Their prescription is as follows. Introduce a copy of the bosons $a(t)$ ($t \in \mathbb{R}$) satisfying the relation $[a(t), b(t')] = 0$ and the same commutation relation as for $b(t)$. Let

$$\tilde{b}(t) = \frac{b(t)}{1 - \mathrm{e}^{-\lambda t}} + a(-t) \qquad \tilde{b}(-t) = \frac{a(t)}{\mathrm{e}^{\lambda t} - 1} + b(-t) \qquad (t > 0). \tag{6.6}$$

Let us introduce the Fock space $\mathcal{F}_a$ generated by the vacuum vector $|vac\rangle_a$ which satisfies

$$a(t)|vac\rangle_a = 0 \qquad \text{if} \quad t > 0. \tag{6.7}$$

For a linear operator $\mathcal{O}$ on the Fock space $\mathcal{F}$, let $\tilde{\mathcal{O}}$ be the operator on $\mathcal{F} \otimes \mathcal{F}_a$ obtained by substituting $\tilde{b}(t)$ for $b(t)$. We then have

$$\frac{\mathrm{tr}_{\mathcal{F}}\left(\mathrm{e}^{-\lambda D}\mathcal{O}\right)}{\mathrm{tr}_{\mathcal{F}}(\mathrm{e}^{-\lambda D})} = \langle\widetilde{0}|\tilde{\mathcal{O}}|\widetilde{0}\rangle \tag{6.8}$$

where the left-hand side denotes the usual expectation value with respect to the Fock vacuum,

$$|\widetilde{0}\rangle = |vac\rangle \otimes |vac\rangle_a \qquad \langle\widetilde{0}|\widetilde{0}\rangle = 1. \tag{6.9}$$

In what follows we use the following abbreviation:

$$\langle \mathcal{O} \rangle_\lambda = \frac{\mathrm{tr}_{\mathcal{F}}\left(\mathrm{e}^{-\lambda D}\mathcal{O}\right)}{\mathrm{tr}_{\mathcal{F}}(\mathrm{e}^{-\lambda D})}. \tag{6.10}$$

We have the following:

$$\langle b_j(t) b_k(t')\rangle_\lambda = \frac{\mathrm{e}^{\lambda t}}{\mathrm{e}^{\lambda t}-1}[b_j(t), b_k(t')]. \tag{6.11}$$

We have the following formula which is useful for evaluating the function (6.5):

$$\frac{\mathrm{tr}_{\mathcal{F}}\left(\mathrm{e}^{-\lambda D} : \exp\left\{\int_{-\infty}^{\infty} c(t)\,\mathrm{e}^{\mathrm{i}\beta_1 t}\,\mathrm{d}t\right\} : : \exp\left\{\int_{-\infty}^{\infty} \mathrm{d}(t)\,\mathrm{e}^{\mathrm{i}\beta_1 t}\mathrm{d}t\right\} :\right)}{\mathrm{tr}_{\mathcal{F}}(\mathrm{e}^{-\lambda D})}$$

$$= \exp\left(\int_0^\infty A(t)\frac{\mathrm{ch}(\mathrm{i}(\beta_1-\beta_2)+\frac{\lambda}{2})t}{\mathrm{sh}\,\frac{\lambda t}{2}}\mathrm{d}t\right) \tag{6.12}$$

where $c(t)$ and $d(t)$ are bosons satisfying $[c(t), d(t')] = A(t)\delta(t+t')$ and $A(t) = -A(-t)$. In order to understand the integral on the right-hand side, see appendix B.

The basic trace functions are evaluated as follows:

$$\langle U_0(\alpha_1)U_0(\alpha_2)\rangle_\lambda = \langle U_n(\alpha_1)U_n(\alpha_2)\rangle_\lambda = \text{constant} \times E_\lambda(\alpha_1-\alpha_2) \tag{6.13}$$

$$\langle U_0(\alpha_1)U_n(\alpha_2)\rangle_\lambda = \langle U_n(\alpha_1)U_0(\alpha_2)\rangle_\lambda = \text{constant} \times E^*_\lambda(\alpha_1-\alpha_2). \tag{6.14}$$

Here we set

$$E_\lambda(\alpha) = \frac{S_3(-\mathrm{i}\alpha+\frac{2\pi}{n})S_3(\mathrm{i}\alpha+\frac{2\pi}{n}+\lambda)}{S_3(-\mathrm{i}\alpha+\frac{2\pi}{n}\xi)S_3(\mathrm{i}\alpha+\frac{2\pi}{n}\xi+\lambda)} \tag{6.15}$$

$$E^*_\lambda(\alpha) = \frac{S_3(-\mathrm{i}\alpha+\pi)S_3(\mathrm{i}\alpha+\lambda+\pi)}{S_3(-\mathrm{i}\alpha+\frac{2\pi}{n}+\pi)S_3(\mathrm{i}\alpha+\lambda+\pi+\frac{2\pi}{n})} \tag{6.16}$$

where

$$S_3(\beta) = S_3\left(\beta \,\middle|\, 2\pi, \frac{2\pi}{n}\xi, \lambda\right) \tag{6.17}$$

$$\langle U_j(\alpha_1)U_{j-1}(\alpha_2)\rangle_\lambda = \langle U_{j-1}(\alpha_1)U_j(\alpha_2)\rangle_\lambda$$

$$= \text{constant} \times \varphi(\alpha_1-\alpha_2)\frac{1}{\mathrm{sh}\left(\frac{n}{2\xi}(\alpha_1-\alpha_2-\frac{\pi\mathrm{i}}{n}+\lambda\mathrm{i})\right)} \tag{6.18}$$

$$\langle U_j(\alpha_1)U_j(\alpha_2)\rangle_\lambda = \text{constant} \times \psi(\alpha_1-\alpha_2)\,\mathrm{sh}\left(\frac{n}{2\xi}\alpha\right)\mathrm{sh}\left(\frac{n}{2\xi}\left(\alpha+\frac{2\pi\mathrm{i}}{n}\right)\right)$$

$$\times\,\mathrm{sh}\left(\frac{\pi}{\lambda}\left(\alpha+\frac{2\pi\mathrm{i}}{n}\right)\right)\mathrm{sh}\left(\frac{\pi}{\lambda}\left(-\alpha+\frac{2\pi\mathrm{i}}{n}\right)\right). \tag{6.19}$$

Here we set

$$\varphi(\alpha) = \frac{1}{S_2\left(\mathrm{i}\alpha+\frac{\pi}{n}\,\middle|\,\lambda, \frac{2\pi}{n}\xi\right)S_2\left(-\mathrm{i}\alpha+\frac{\pi}{n}\,\middle|\,\lambda, \frac{2\pi}{n}\xi\right)} \tag{6.20}$$

$$\psi(\alpha) = \frac{1}{S_2\left(\mathrm{i}\alpha-\frac{2\pi}{n}\,\middle|\,\lambda, \frac{2\pi}{n}\xi\right)S_2\left(-\mathrm{i}\alpha-\frac{2\pi}{n}\,\middle|\,\lambda, \frac{2\pi}{n}\xi\right)}. \tag{6.21}$$

The functions $\varphi(\alpha)$ and $\psi(\alpha)$ become the integral kernel of the correlation functions.

The function $\varphi(\alpha)$ has poles at

$$\alpha = \pm\,\mathrm{i}\left(n_1\lambda + n_2\frac{2\pi}{n}\xi + \frac{\pi}{n}\right) \qquad (n_1, n_2 \geqslant 0). \tag{6.22}$$

The function $\psi(\alpha)$ has poles at

$$\alpha = \pm\,\mathrm{i}\left(n_1\lambda + n_2\frac{2\pi}{n}\xi - \frac{2\pi}{n}\right) \qquad (n_1, n_2 \geqslant 0). \tag{6.23}$$

The trace of the vertex operators (6.5) is evaluated by applying Wick's theorem.

In what follows we use

$$\rho = \frac{n}{2\xi}. \tag{6.24}$$

The two-point correlation functions are evaluated as follows:

$$\begin{aligned} G^{(1)}(\beta_1|\beta_1')_{\epsilon,\epsilon} = {} & E_\lambda^*(\beta_1 - \beta_1')\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_1}{2\pi\mathrm{i}}\cdots\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_{n-1}}{2\pi\mathrm{i}}\prod_{k=1}^{n}\varphi(\alpha_k - \alpha_{k-1})\,\mathrm{e}^{\rho(\alpha_\epsilon + \alpha_{\epsilon+1} - \beta_1 - \beta_1')} \\ & \times \mathrm{sh}\left(\rho\left(\alpha_{\epsilon+1} - \alpha_\epsilon - \frac{\pi\mathrm{i}}{n}\right)\right) \\ & \times \prod_{k=1}^{n}\frac{1}{\mathrm{sh}\left(\rho(\alpha_k - \alpha_{k-1} - \frac{\pi\mathrm{i}}{n})\right)\mathrm{sh}\left(\rho(\alpha_k - \alpha_{k-1} - \frac{\pi\mathrm{i}}{n} + \lambda\mathrm{i})\right)} \end{aligned} \tag{6.25}$$

where $\alpha_n = \beta_1$ and $\alpha_0 = \beta_1'$. Here we omit an irrelevant constant factor.

The $2N$-point correlation functions are evaluated as follows:

$$\begin{aligned} & G^{(N)}(\beta_1\cdots\beta_N|\beta_N'\cdots\beta_1')_{\epsilon_1\cdots\epsilon_N,\epsilon_N,\ldots,\epsilon_1} \\ & \quad = E_\lambda(\beta_1\cdots\beta_N|\beta_N'\cdots\beta_1')\prod_{j,r}\int_{-\infty}^{\infty}\mathrm{d}\alpha_{j,r}\,K_\lambda(\{\alpha_{j,r}\})I_{\lambda;\epsilon_1\cdots\epsilon_N}(\{\alpha_{j,r}\}). \end{aligned} \tag{6.26}$$

We associate the variables $\alpha_{j,r}$ $(1 \leqslant r \leqslant N, 0 \leqslant j \leqslant \epsilon_r)$ with the basic operator $U_j(\alpha_{j,r})$ contained in the vertex operators $\Phi_{\epsilon_r}(\beta_r')$ and the variables $\alpha_{j,r}$ $(1 \leqslant r \leqslant N, \epsilon_r + 1 \leqslant j \leqslant n)$ with the basic operator $U_j(\alpha_{j,r})$ contained in the vertex operators $\Phi^*_{\epsilon_r}(\beta_r)$. The number of integrals $\prod_{r=1}^{N}\prod_{j=1}^{n-1}\int\mathrm{d}\alpha_{j,r}$ is $N(n-1)$.

We set the index sets $\mathcal{N}_j^*$ and $\mathcal{N}_j$ as follows:

$$\alpha_{n,r} = \beta_r \qquad \alpha_{0,r} = \beta_r' \tag{6.27}$$

$$\mathcal{N}_j^* = \{k|\epsilon_k \leqslant j-1\} \qquad \mathcal{N}_j = \{k|\epsilon_k \geqslant j\}. \tag{6.28}$$

The function $E_\lambda(\beta_1\cdots\beta_N|\beta_N'\cdots\beta_1')$ is given by

$$\begin{aligned} E_\lambda(\beta_1\cdots\beta_N|\beta_N'\cdots\beta_1') = {} & \mathrm{e}^{-\rho(\beta_1+\cdots+\beta_N+\beta_1'+\cdots+\beta_N')} \\ & \times \prod_{1\leqslant j<k\leqslant N}E_\lambda(\beta_j - \beta_k)\prod_{1\leqslant j<k\leqslant N}E_\lambda(\beta_k' - \beta_j')\prod_{j,k=1}^{N}E_\lambda^*(\beta_j - \beta_k'). \end{aligned} \tag{6.29}$$

The integral kernel $K_\lambda(\{\alpha_{j,r}\})$ is given by

$$\begin{aligned} K_\lambda(\{\alpha_{j,r}\}) = {} & \prod_{j=1}^{n-1}\prod_{\substack{r,s=1\\ r>s}}^{N}\psi(\alpha_{j,r} - \alpha_{j,s}) \times \prod_{j=1}^{n-1}\prod_{k,l=1}^{N}\left\{\prod_{\substack{k\in N_j^*\\ l\in N_{j-1}}}\varphi(\alpha_{j,k} - \alpha_{j,l})\prod_{\substack{k\in N_{j-1}^*\\ l\in N_j}}\varphi(\alpha_{j,k} - \alpha_{j,l})\right\} \\ & \times \prod_{j=1}^{n-1}\prod_{k,l=1}^{N}\left\{\prod_{\substack{k\in N_j\\ l\in N_{j-1}}}\varphi(\alpha_{j,k} - \alpha_{j,l})\prod_{\substack{k\in N_j^*\\ l\in N_{j-1}^*}}\varphi(\alpha_{j,k} - \alpha_{j,l})\right\}^2. \end{aligned} \tag{6.30}$$

The integrand $I_{\lambda;\epsilon_1\cdots\epsilon_N}(\{\alpha_{j,r}\})$ is given by

$$I_{\lambda;\epsilon_1\cdots\epsilon_N}(\{\alpha_{j,r}\}) = g_\lambda(\{\alpha_{j,r}\})h_\lambda(\{\alpha_{j,r}\}). \tag{6.31}$$

Here we set

$$h_\lambda(\{\alpha_{j,r}\}) = \prod_{j=1}^{n-1}\prod_{\substack{r,s=1\\ r>s}}^{N}\left\{\mathrm{sh}\left(\frac{\pi}{\lambda}(\alpha_{j,r}-\alpha_{j,s}+\frac{2\pi\mathrm{i}}{n})\right)\mathrm{sh}\left(\frac{\pi}{\lambda}(\alpha_{j,s}-\alpha_{j,r}+\frac{2\pi\mathrm{i}}{n})\right)\right\} \tag{6.32}$$

and

$$\begin{aligned}
g_\lambda(\{\alpha_{j,r}\}) = &\prod_{j=1}^{n-1}\prod_{\substack{r>s\\ s\in N_j}}\left\{\mathrm{sh}\left(\rho(\alpha_{j,r}-\alpha_{j,s})\right)\mathrm{sh}\left(\rho(\alpha_{j,r}-\alpha_{j,s}+\frac{2\pi\mathrm{i}}{n})\right)\right\}\\
&\times\prod_{j=1}^{n-1}\prod_{\substack{r>s\\ s\in N_j^*}}\left\{\mathrm{sh}\left(\rho(\alpha_{j,s}-\alpha_{j,r})\right)\mathrm{sh}\left(\rho(\alpha_{j,s}-\alpha_{j,r}+\frac{2\pi\mathrm{i}}{n})\right)\right\}\\
&\times\prod_{j=1}^{n}\prod_{\substack{r\in N_j\\ s\in N_{j-1}}}\left\{\mathrm{sh}\left(\rho(\alpha_{j,r}-\alpha_{j-1,s}-\frac{\pi\mathrm{i}}{n}+\lambda\mathrm{i})\right)\right.\\
&\left.\times\,\mathrm{sh}\left(\rho(\alpha_{j-1,r}-\alpha_{j,s}-\frac{\pi\mathrm{i}}{n}+\lambda\mathrm{i})\right)\right\}^{-1}\\
&\times\prod_{j=1}^{n}\prod_{\substack{r\in N_j^*\\ s\in N_{j-1}^*}}\left\{\mathrm{sh}\left(\rho(\alpha_{j,r}-\alpha_{j-1,s}-\frac{\pi\mathrm{i}}{n}+\lambda\mathrm{i})\right)\right.\\
&\left.\times\,\mathrm{sh}\left(\rho(\alpha_{j-1,r}-\alpha_{j,s}-\frac{\pi\mathrm{i}}{n}+\lambda\mathrm{i})\right)\right\}^{-1}\\
&\times\prod_{j=1}^{n}\left\{\prod_{\substack{r\in N_j^*\\ s\in N_{j-1}}}\mathrm{sh}\left(\rho(\alpha_{j,r}-\alpha_{j-1,s}-\frac{\pi\mathrm{i}}{n}+\lambda\mathrm{i})\right)\right.\\
&\left.\times\prod_{\substack{r\in N_{j-1}^*\\ s\in N_j}}\mathrm{sh}\left(\rho(\alpha_{j,r}-\alpha_{j-1,s}-\frac{\pi\mathrm{i}}{n}+\lambda\mathrm{i})\right)\right\}^{-1}\\
&\times\prod_{r=1}^{N}\mathrm{e}^{\rho(\alpha_{\epsilon_r}+\alpha_{\epsilon_r+1})}\prod_{r=1}^{N}\mathrm{sh}\left(\rho(\alpha_{\epsilon_r+1,r}-\alpha_{\epsilon_r,r}-\frac{\pi\mathrm{i}}{n})\right)\\
&\times\prod_{j=1}^{n}\prod_{r=1}^{N}\left\{\mathrm{sh}\left(\rho(\alpha_{j,r}-\alpha_{j-1,r}-\frac{\pi\mathrm{i}}{n})\right)\right\}^{-1}.
\end{aligned} \tag{6.33}$$

Here we omit an irrelevant constant factor.

Now we state a short summary of this paper. In this paper we have proposed a free-field construction of correlation functions. We presented the free boson realizations of the vertex operators (4.10) and (4.11). Using those realizations, we derived integral representations (6.26) for the correlation functions $G^{(N)}(\beta_1\cdots\beta_N|\beta'_N\cdots\beta'_1)$. *Here the leading factor*

$E_\lambda(\beta_1\cdots\beta_N|\beta'_N\cdots\beta'_1)$, the integral kernel $K_\lambda(\{\alpha_{j,r}\})$ and the integrand $I_{\lambda;\epsilon_1\cdots\epsilon_N}(\{\alpha_{j,r}\})$ are given in (6.29)–(6.31).

We give a short discussion of topics relating to our problem. For the special case $n=2$ and $\lambda=2\pi$, Jimbo and Miwa [7] reduced the N-fold integral for $G^{(N)}(\beta_1\cdots\beta_N|\beta_{N+1}\cdots\beta_{2N})$ to an $(N-1)$-fold integral by carrying out the integration once. In the future we intend to reduce the number of integrals of our integral representations for a higher-rank model, and obtain a higher-rank generalization of [7]. For another application of the free boson realization of the vertex operators given in this paper, the critical $A^{(1)}_{n-1}$ chain with a boundary is our future problem. For this purpose the references [17, 18] are useful.

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Appendix A. Multi-gamma functions

Here we summarize the multiple gamma and the multiple sine functions. Let us set the functions $\Gamma_1(x|\omega)$, $\Gamma_2(x|\omega_1,\omega_2)$ and $\Gamma_3(x|\omega_1,\omega_2,\omega_3)$ by

$$\log\Gamma_1(x|\omega)+\gamma B_{11}(x|\omega)=\int_C\frac{\mathrm{d}t}{2\pi\mathrm{i}t}\mathrm{e}^{-xt}\frac{\log(-t)}{1-\mathrm{e}^{-\omega t}} \tag{A.1}$$

$$\log\Gamma_2(x|\omega_1,\omega_2)-\frac{\gamma}{2}B_{22}(x|\omega_1,\omega_2)=\int_C\frac{\mathrm{d}t}{2\pi\mathrm{i}t}\mathrm{e}^{-xt}\frac{\log(-t)}{(1-\mathrm{e}^{-\omega_1 t})(1-\mathrm{e}^{-\omega_2 t})} \tag{A.2}$$

$$\log\Gamma_3(x|\omega_1,\omega_2,\omega_3)+\frac{\gamma}{3!}B_{33}(x|\omega_1,\omega_2,\omega_3)$$
$$=\int_C\frac{\mathrm{d}t}{2\pi\mathrm{i}t}\mathrm{e}^{-xt}\frac{\log(-t)}{(1-\mathrm{e}^{-\omega_1 t})(1-\mathrm{e}^{-\omega_2 t})(1-\mathrm{e}^{-\omega_3 t})} \tag{A.3}$$

where the functions $B_{jj}(x)$ are the multiple Bernoulli polynomials defined by

$$\frac{t^r\mathrm{e}^{xt}}{\prod_{j=1}^r(\mathrm{e}^{\omega_j t}-1)}=\sum_{n=0}^\infty\frac{t^n}{n!}B_{r,n}(x|\omega_1\cdots\omega_r) \tag{A.4}$$

or more explicitly

$$B_{11}(x|\omega)=\frac{x}{\omega}-\frac{1}{2} \tag{A.5}$$

$$B_{22}(x|\omega)=\frac{x^2}{\omega_1\omega_2}-\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)x+\frac{1}{2}+\frac{1}{6}\left(\frac{\omega_1}{\omega_2}+\frac{\omega_2}{\omega_1}\right). \tag{A.6}$$

Here γ is Euler's constant, $\gamma=\lim_{n\to\infty}(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n)$.

Here the contour of integral is given in figure A1.

Let us set

$$S_1(x|\omega)=\frac{1}{\Gamma_1(\omega-x|\omega)\Gamma_1(x|\omega)} \tag{A.7}$$

$$S_2(x|\omega_1,\omega_2)=\frac{\Gamma_2(\omega_1+\omega_2-x|\omega_1,\omega_2)}{\Gamma_2(x|\omega_1,\omega_2)} \tag{A.8}$$

$$S_3(x|\omega_1,\omega_2,\omega_3)=\frac{1}{\Gamma_3(\omega_1+\omega_2+\omega_3-x|\omega_1,\omega_2,\omega_3)\Gamma_3(x|\omega_1,\omega_2,\omega_3)} \tag{A.9}$$

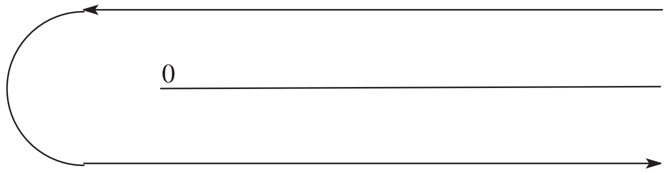

Figure A1. Contour C.

We have

$$\Gamma_1(x|\omega) = \mathrm{e}^{(\frac{x}{\omega}-\frac{1}{2})\log\omega}\frac{\Gamma(x/\omega)}{\sqrt{2\pi}} \qquad S_1(x|\omega) = 2\sin(\pi x/\omega) \tag{A.10}$$

$$\frac{\Gamma_2(x+\omega_1|\omega_1,\omega_2)}{\Gamma_2(x|\omega_1,\omega_2)} = \frac{1}{\Gamma_1(x|\omega_2)} \qquad \frac{S_2(x+\omega_1|\omega_1,\omega_2)}{S_2(x|\omega_1,\omega_2)} = \frac{1}{S_1(x|\omega_2)} \qquad \frac{\Gamma_1(x+\omega|\omega)}{\Gamma_1(x|\omega)} = x \tag{A.11}$$

$$\frac{\Gamma_3(x+\omega_1|\omega_1,\omega_2,\omega_3}{\Gamma_3(x|\omega_1,\omega_2,\omega_3)} = \frac{1}{\Gamma_2(x|\omega_2,\omega_3)} \qquad \frac{S_3(x+\omega_1|\omega_1,\omega_2,\omega_3)}{S_3(x|\omega_1,\omega_2,\omega_3)} = \frac{1}{S_2(x|\omega_2,\omega_3)} \tag{A.12}$$

$$\log S_2(x|\omega_1\omega_2) = \int_C \frac{\mathrm{sh}(x-\frac{\omega_1+\omega_2}{2})t}{2\,\mathrm{sh}\,\frac{\omega_1 t}{2}\,\mathrm{sh}\,\frac{\omega_2 t}{2}}\log(-t)\frac{\mathrm{d}t}{2\pi\mathrm{i}t} \qquad (0 < \mathrm{Re}\,x < \omega_1+\omega_2) \tag{A.13}$$

$$S_2(x|\omega_1\omega_2) = \frac{2\pi}{\sqrt{\omega_1\omega_2}}x + \mathrm{O}(x^2) \qquad (x \to 0). \tag{A.14}$$

$$S_2(x|\omega_1\omega_2)S_2(-x|\omega_1\omega_2) = -4\sin\frac{\pi x}{\omega_1}\sin\frac{\pi x}{\omega_2}. \tag{A.15}$$

Appendix B. Normal ordering

Here we list the formulae of the form

$$X(\beta_1)Y(\beta_2) = C_{XY}(\beta_1-\beta_2) : X(\beta_1)X(\beta_2) : \tag{B.1}$$

where $X, Y = U_j$ and $C_{XY}(\beta)$ is a meromorphic function on $\mathbb{C}$. These formulae follow from the commutation relation of the free bosons. When we compute the contraction of the basic operators, we often encounter an integral

$$\int_0^\infty F(t)\,\mathrm{d}t \tag{B.2}$$

which is divergent at $t = 0$. Here we adopt the following prescription for regularization: it should be understood as the contour integral,

$$\int_C F(t)\frac{\log(-t)}{2\pi\mathrm{i}}\,\mathrm{d}t. \tag{B.3}$$

Here we use the same contour C as given in appendix A.

The contractions of the basic operators $U_j(\beta)$ have the following formulae:

$$U_{j_1}(\alpha_1)U_{j_2}(\alpha_2) = h_{j_1,j_2}(\alpha_1-\alpha_2) : U_{j_2}(\alpha_2)U_{j_1}(\alpha_1) : \qquad (0 \leqslant j_1, j_2 \leqslant n). \tag{B.4}$$

Here we have set the multiplicity functions $h_{j_1,j_2}(\alpha)$ as

$$h_{0,0}(\alpha) = h_{n,n}(\alpha) = \exp\left\{\gamma\frac{\xi-1}{\xi}\frac{n-1}{n}\right\}$$
$$\times\frac{\Gamma_2(-\mathrm{i}\alpha+\frac{2\pi}{n}\xi|\frac{2\pi}{n}\xi,2\pi)\Gamma_2(-\mathrm{i}\alpha+2\pi|\frac{2\pi}{n}\xi,2\pi)}{\Gamma_2(-\mathrm{i}\alpha+\frac{2\pi}{n}|\frac{2\pi}{n}\xi,2\pi)\Gamma_2(-\mathrm{i}\alpha+2\pi+\frac{2\pi}{n}\xi-\frac{2\pi}{n}|\frac{2\pi}{n}\xi,2\pi)} \quad \text{(B.5)}$$

$$h_{0,n}(\alpha) = h_{n,0}(\alpha) = \exp\left\{\frac{\gamma}{n}\frac{\xi-1}{\xi}\right\}$$
$$\times\frac{\Gamma_2(-\mathrm{i}\alpha+\pi+\frac{2\pi}{n}\xi-\frac{2\pi}{n}|\frac{2\pi}{n}\xi,2\pi)\Gamma_2(-\mathrm{i}\alpha+\pi+\frac{2\pi}{n}|\frac{2\pi}{n}\xi,2\pi)}{\Gamma_2(-\mathrm{i}\alpha+\pi|\frac{2\pi}{n}\xi,2\pi)\Gamma_2(-\mathrm{i}\alpha+\pi+\frac{2\pi}{n}\xi|\frac{2\pi}{n}\xi,2\pi)} \quad \text{(B.6)}$$

and

$$h_{jj}(\alpha) = \mathrm{e}^{2\gamma\frac{\xi-1}{\xi}}\frac{\alpha}{\mathrm{i}}\frac{\Gamma_1(-\mathrm{i}\alpha+\frac{2\pi}{n}\xi-\frac{2\pi}{n}|\frac{2\pi}{n}\xi)}{\Gamma_1(-\mathrm{i}\alpha+\frac{2\pi}{n}|\frac{2\pi}{n}\xi)} \qquad (1\leqslant j\leqslant n-1) \quad \text{(B.7)}$$

$$h_{jj-1}(\alpha) = \mathrm{e}^{-\gamma\frac{\xi-1}{\xi}}\frac{\Gamma_1(-\mathrm{i}\alpha+\frac{\pi}{n}|\frac{2\pi}{n}\xi)}{\Gamma_1(-\mathrm{i}\alpha+\frac{\pi}{n}(2\xi-1)|\frac{2\pi}{n}\xi)} \qquad (1\leqslant j\leqslant n). \quad \text{(B.8)}$$

Other entries of $h_{j_1,j_2}(\alpha)$ are zero.

The commutation relations of the basic operators $U_j(\beta)$ have the following formulae:

$$U_{j_1}(\beta_1)U_{j_2}(\beta_2) = H_{j_1,j_2}(\beta_1-\beta_2)U_{j_2}(\beta_2)U_{j_1}(\beta_1). \quad \text{(B.9)}$$

Here we have set the functions $H_{j_1,j_2}(\beta)$ as

$$H_{j,j}(\beta) = -\frac{\mathrm{sh}\left(\frac{n}{2\xi}(\beta+\frac{2\pi\mathrm{i}}{n})\right)}{\mathrm{sh}\left(\frac{n}{2\xi}(-\beta+\frac{2\pi\mathrm{i}}{n})\right)} \qquad (j=1,\ldots,n-1) \quad \text{(B.10)}$$

$$H_{j,j-1}(\beta) = H_{j-1,j}(\beta) = \frac{\mathrm{sh}\left(\frac{n}{2\xi}\left(\beta-\frac{\pi\mathrm{i}}{n}\right)\right)}{\mathrm{sh}\left(\frac{n}{2\xi}\left(-\beta-\frac{\pi\mathrm{i}}{n}\right)\right)} \qquad (j=1,\ldots,n) \quad \text{(B.11)}$$

$$H_{0,0}(\beta) = H_{n,n}(\beta) = -\frac{S_2(\mathrm{i}\alpha|\frac{2\pi}{n}\xi,2\pi)S_2(-\mathrm{i}\alpha+\frac{2\pi}{n}|\frac{2\pi}{n}\xi,2\pi)}{S_2(-\mathrm{i}\alpha|\frac{2\pi}{n}\xi,2\pi)S_2(\mathrm{i}\alpha+\frac{2\pi}{n}|\frac{2\pi}{n}\xi,2\pi)} = r(\beta) \quad \text{(B.12)}$$

$$H_{0,n}(\beta) = H_{n,0}(\beta) = \frac{S_2(-\mathrm{i}\alpha+\pi|\frac{2\pi}{n}\xi,2\pi)S_2(\mathrm{i}\alpha+\pi+\frac{2\pi}{n}|\frac{2\pi}{n}\xi,2\pi)}{S_2(\mathrm{i}\alpha+\pi|\frac{2\pi}{n}\xi,2\pi)S_2(-\mathrm{i}\alpha+\pi+\frac{2\pi}{n}|\frac{2\pi}{n}\xi,2\pi)} = r^*(\beta) \quad \text{(B.13)}$$

where $r(\beta)$ is the same as that in (2.1), and $r^*(\beta)$ is the same as that in (3.6). Other entries of $H_{j_1,j_2}(\beta)$ are zero.

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